

## Optimal decisions on the instantaneous rate of growth of consumption in excess of habit and money demand

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### Abstract

**Objective:** this paper develops a time-continuous deterministic model of a rational consumer that maximizes his utility of the instantaneous rate of growth of consumption in excess of habit and real money balances. **Methodology:** we introduce the concept of utility of the instantaneous rate of growth of consumption in excess of habit. The necessary conditions for an interior solution lead to a Bernoulli's differential equation of second order. **Results:** closed form solutions of the instantaneous rate of growth of consumption in excess of habit, consumption, and real money balances are provided. Moreover, economic welfare of the consumer is computed and comparative statics exercises are carried out. **Novelty:** this the first time that in the utility function is included the growth rate of consumption in excess of habit, which provides richer environments for studying rational behavior of habit formation. **Conclusions:** under the assumption of logarithmic utility, it is found that consumption increases linearly with time due to the habit, while money demand remains constant to finance the habit through time.

**Keywords:** consumer behavior, habit persistence, real money balances.

**JEL classification:** D11, P46.

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# Decisiones óptimas sobre la tasa instantánea de crecimiento del consumo en exceso del hábito y la demanda de dinero

## Resumen

**Objetivo:** se desarrolla un modelo determinista en tiempo continuo de un consumidor racional que maximiza utilidad por la tasa de crecimiento instantáneo del consumo en exceso del hábito y los saldos monetarios reales. **Metodología:** se introduce el concepto de utilidad de la tasa de crecimiento instantáneo del consumo en exceso del hábito. Las condiciones necesarias para una solución interna conducen a una ecuación diferencial de Bernoulli de segundo orden. **Resultados:** se proporcionan soluciones en forma cerrada de la tasa de crecimiento instantáneo del consumo en exceso del hábito, el consumo y la demanda de dinero. También se calcula el bienestar económico del consumidor y se realizan ejercicios de estática comparativa. **Originalidad:** es la primera vez que en la función de utilidad se incluye la tasa de crecimiento del consumo en exceso del hábito, lo cual proporciona ambientes más ricos para estudiar el comportamiento racional de la formación del hábito. **Conclusiones:** bajo el supuesto de la utilidad logarítmica se encuentra que el consumo aumenta linealmente con el tiempo como consecuencia del hábito, mientras que la demanda de dinero se mantiene constante para financiar el hábito a través del tiempo.

**Palabras clave:** comportamiento del consumidor, persistencia del hábito, saldos monetarios reales.

**Clasificación JEL:** D11, P46.

## 1. Introduction

Habit persistence, or habit formation, is a preference specification that has long been studied within the framework of a rational consumer. When modeling habit persistence, utility functions derived from the “Bergson family” (see Samuelson, 1965) are widely used. A simple one-period habit formation utility function according to Pollak (1970), Sundaresan (1989), Abel (1990), Galí (1994), and others, is given by:

$$u(c_t, c_{t-1}) \equiv u(c_t - \alpha c_{t-1}), u' > 0, u'' < 0, u \in C^2(0, \infty)$$

where  $c_t$  is current consumption,  $c_{t-1}$  is consumption in the previous period, and  $\alpha \in (0,1]$  is the intensity parameter of habit formation.<sup>1</sup> By choosing  $\alpha=1$ , that is, when consumption in the previous period does not forget habit excess, then we obtain a utility function of excess of habit of the form:

$$u(c_t - c_{t-1}) \equiv u\left(\frac{c_t - c_{t-1}}{t - (t-1)}\right)$$

if instead of considering periods of length one of excess of habit, we consider a time interval of the form  $[t-\Delta t, t]$ , then the utility function of excess of habit during that interval is:

$$u\left(\frac{c_t - c_{t-\Delta t}}{t - (t-\Delta t)}\right) = u\left(\frac{c_t - c_{t-\Delta t}}{\Delta t}\right) \quad (1)$$

By taking the limit as  $\Delta t \rightarrow 0$  in the above equation and using the fact that  $u$  is a continuous function, we get a utility function of the instantaneous rate of growth of consumption in excess of habit  $u(\dot{c}_t)$ . This utility function can be used in the rational choice theory to study the behavior of an individual in terms of the growth rate of consumption in excess of habit, which is useful when consumption in the previous instant fully remembers habit excess.

We adjoin utility of money to the utility of the instantaneous rate of growth of consumption in excess of habit to examine the relationship between the optimal path of consumption and the optimal money demand. Specifically, we want to see if they follow similar dynamics, as in the case typical case of  $u(c_t) = \ln(c_t)$ , where the paths differ only from the opportunity cost of holding real money balances. Finally, we will investigate the effect on consumption of the opportunity cost of holding real money balances as time passes by.

The paper is organized as follow: in the next section, we set up the utility maximization problem of the rational consumer considering the instantaneous rate of growth of consumption in excess of habit and real money balances; section 3 deals with the first order conditions for interior solutions; section 4 determines closed-form solutions of optimal decisions; section 5 computes the economic welfare of the rational consumer and comparative statics exercises are carried out; finally, section 6 presents conclusions and recognizes the delimitations.

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<sup>1</sup> See also Constantinides (1990), Ferson and Constantinides (1991), and Li (2001).

## 2. Utility maximization of instantaneous excess of habit and real money

In this section, we state a continuous time, deterministic model of a competitive, infinitely-lived rational consumer that maximizes his utility of the instantaneous rate of growth of consumption in excess of habit and real money balances:

$$\begin{aligned} & \text{Maximize } \int_0^{\infty} [\ln(\dot{c}_t) + \theta \ln(m_t)] e^{-\rho t} dt \\ & \text{subject to} \\ & b_0 = \int_0^{\infty} (\dot{m}_t + \pi m_t + c_t) e^{-rt} dt \end{aligned}$$

where  $\dot{c}_t$  is the instantaneous rate of growth of consumption in excess of habit,  $m_t$  stands for real money balances at time  $t$ ,  $r > 0$  denotes a continuously compound real interest rate,  $\rho > 0$  is the subjective discount rate,  $\theta > 0$  is a deep parameter that measures the individual's relative significance of holding money,  $b_0$  is the initial real wealth, and  $\pi > 0$  is the inflation rate. Notice that  $(\dot{c}_t)$ , must remain positive, so utility on consumption,  $\ln(\dot{c}_t)$ , is well defined. Moreover, the integral of  $c_t$  must also remain positive.

## 3. First order conditions for utility maximization

In this case, the Lagrangean associated with the previous problem is given by:

$$\mathcal{L}(c_t, \dot{c}_t, m_t, \dot{m}_t, \lambda) = [\ln(\dot{c}_t) + \theta \ln(m_t)] e^{-\rho t} - \lambda (\dot{m}_t + \pi m_t + c_t) e^{-rt}$$

where the multiplier  $\lambda > 0$  has to be determined. The Euler–Lagrange equations of the calculus of variations are given by:<sup>2</sup>

$$\frac{\partial \mathcal{L}}{\partial c_t} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{c}_t} = 0$$

and

$$\frac{\partial \mathcal{L}}{\partial m_t} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{m}_t} = 0$$

equivalently,

$$\lambda e^{-rt} + \frac{\ddot{c}_t}{\dot{c}_t^2} e^{-\rho t} + \frac{\rho}{\dot{c}_t} e^{-\rho t} = 0 \quad (2)$$

<sup>2</sup> Regarding the Euler-Lagrange conditions, see for instance, Venegas-Martínez (2008).

and

$$\frac{\theta}{m_t} e^{-\rho t} - \lambda(\pi + r) e^{-rt} = 0 \quad (3)$$

from equation (3), we find that

$$m_t = \frac{\theta}{i\lambda} e^{(r-\rho)t} \quad (4)$$

where  $i = r + \pi$  is the nominal interest rate. From the assumptions  $r > 0$  and  $\pi > 0$ , we have  $i > \pi > 0$ . From (2), we obtain the following second-order differential equation in  $c_t$ :

$$\lambda e^{-rt} \dot{c}_t^2 + \ddot{c}_t e^{-\rho t} + \rho e^{-\rho t} \dot{c}_t = 0 \quad (5)$$

The above differential equation belongs to the Bernoulli's family. In order to solve this differential equation, we first define, as usual, the following variable change

$$\dot{c}_t = \frac{1}{u_t} \quad (6)$$

then

$$\ddot{c}_t = -\frac{\dot{u}_t}{u_t^2}$$

therefore, equation (5) becomes

$$\dot{u}_t = \rho u_t - \lambda e^{(\rho-r)t} \quad (7)$$

which is a linear non-homogeneous differential equation of first order. The solution of the above equation is given by,

$$u_t = \left( u_0 - \frac{\lambda}{r} \right) e^{\rho t} + \frac{\lambda}{r} e^{(\rho-r)t}$$

where the initial condition  $u_0$  has to be determined together with  $\lambda$ .

#### 4. Closed-form solutions of optimal decisions

If we take into account (6) in the previous section, we get:

$$\dot{c}_t = \frac{1}{u_t} = \frac{e^{-\rho t}}{u_0 - \frac{\lambda}{r} (1 - e^{-rt})} \quad (8)$$

Notice first that  $u_0=1/(\dot{c}_0)$ , and  $u_0$  must be positive since  $u(\dot{c}_0)=\ln(1/u_0)$  has to make sense. For the sake of simplicity, let us assume, as in Calvo (1986) and Venegas-Martínez (2006) and (2001), that  $r=\rho$ . In other words, the subjective discount rate is equal to the real interest rate; that is, they coincide by chance. This will allow us to obtain a steady state for real money balances,  $(\dot{m}_t)=0$ . This will make easier the task of finding closed solutions; otherwise it is not possible to determine an analytical solution; as we will see latter. By integrating (8), we obtain:

$$c_t = -\int_0^t \frac{-e^{-\rho s} ds}{u_0 - \frac{\lambda}{\rho}(1 - e^{-\rho s})} = -\frac{1}{\lambda} \ln \left( \frac{u_0 - \frac{\lambda}{\rho}(1 - e^{-\rho t})}{u_0} \right) = -\frac{1}{\lambda} \ln \left( 1 - \frac{\lambda(1 - e^{-\rho t})}{\rho u_0} \right) \quad (9)$$

we now have to assume that  $\lambda=\rho u_0$  to assure positive closed-form solutions. This relation among parameters makes sense since both  $\lambda$  and  $u_0$  are to be determined simultaneously from the Euler-Lagrange equation associated with consumption given the change of variable in (6). Therefore,

$$c_t = \frac{\rho t}{\lambda} \quad (10)$$

thus,  $\dot{c}_t=\rho/\lambda$ . Notice first that from (4) and under the assumption  $r=\rho$ , we have

$$m_t = \frac{\theta}{i\lambda} \quad (11)$$

in this case,  $\dot{m}_t=0$ . That is, we obtain a steady state for real money balances. Let us substitute (10) and (11) in the budget constraint in order to determine  $\lambda$ . Therefore,

$$b_0 = \int_0^\infty (\dot{m}_t + \pi m_t + c_t) e^{-\rho t} dt = \int_0^\infty \left[ \frac{\pi\theta}{i\lambda} + \frac{\rho t}{\lambda} \right] e^{-\rho t} dt = \frac{\pi\theta}{i\lambda\rho} + \frac{1}{\lambda\rho}$$

this, now, allows us to solve for  $\lambda$  in terms of exogenous variables. Therefore,

$$\frac{1}{\lambda} = \frac{\rho b_0}{1 + \frac{\pi\theta}{i}} \quad (12)$$

in this case, we find the initial values in terms of exogenous variables,

$$u_0 = \frac{1}{\dot{c}_0} = \frac{\lambda}{\rho} = \frac{1 + \left(\frac{\pi\theta}{i}\right)}{\rho^2 b_0}$$

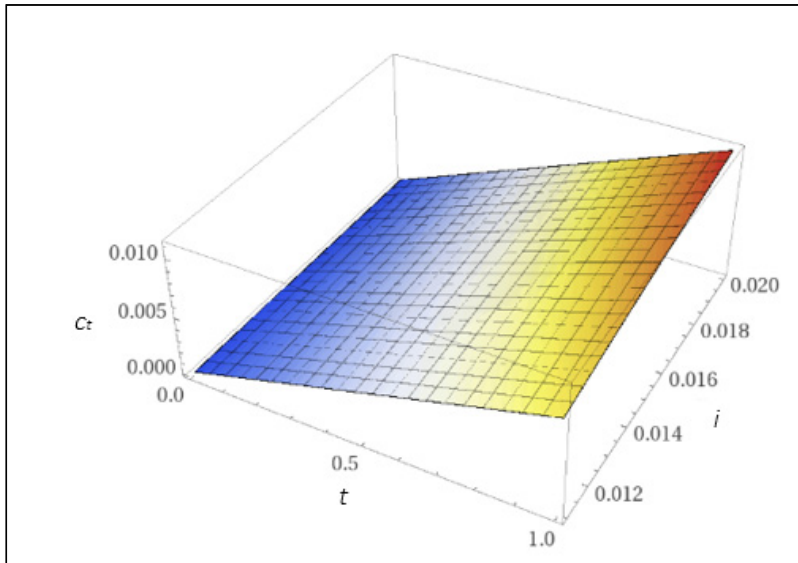
by substituting (12) in (9) and (11), respectively, we obtain closed-form solutions:

$$c_t = \frac{\rho^2 b_0 t}{1 + \frac{\pi \theta}{i}} > 0 \quad (13)$$

and

$$m_t = \frac{\theta \rho b_0}{i + \pi \theta} = \text{constant} > 0 \quad (14)$$

Figure 1 shows the dependence of optimal consumption,  $c_t$ , on the nominal interest rate,  $i$ , as time passes by where; here we have assumed that  $\pi=0.01, \theta=2, b_0=1$  and  $\rho=0.15$ . As it can be observed, consumption increases when both  $t$  and  $i$  grow.



Source: own elaboration with Wolfram Alpha Software

Figure 1  
Consumption as a function of  $i \in (0.011, 0.02)$  and  $t \in (0,1)$

Observe now that consumption increases linearly with time with positive slope:

$$\dot{c}_t = \frac{\rho^2 b_0}{1 + \frac{\pi \theta}{i}} > 0 \quad (15)$$

while money demand remains constant through time. Notice also that a once-and-for-all increase in the nominal interest rate results in a reduction of real money balances. Finally, observe that the factor  $\rho b_0$  in (13) and (14) is a standard

factor in optimal decisions for logarithmic utility, which it also appears now in the instantaneous rate of growth of consumption in excess of habit.

## 5. Economic welfare

In this section, we compute the consumer economic welfare,  $W$ , and carry out some comparative statics exercises. In this case, by taking into account (14) and (15), the indirect utility function satisfies:

$$W = \int_0^\infty \left[ \ln \left( \frac{\rho^2 b_0}{1 + \frac{\pi\theta}{t}} \right) + \theta \ln \left( \frac{\theta b_0}{i + \pi\theta} \right) \right] e^{-\rho t} dt = \frac{1}{\rho} \left[ \ln \left( \frac{\rho^2 b_0}{1 + \frac{\pi\theta}{t}} \right) + \theta \ln \left( \frac{\theta b_0}{i + \pi\theta} \right) \right]$$

Once economic welfare has been computed, we carry out some comparative statics exercises by examining the signs of following partial differential equations:

$$\frac{\partial W}{\partial b_0} > 0, \quad \frac{\partial W}{\partial \pi} < 0, \quad \text{and} \quad \frac{\partial W}{\partial \rho} < 0$$

Hence, an increase in initial real wealth enhances welfare. Also, an increase in inflation results in a reduction of welfare. Finally, the effect of the subjective discount rate on consumption is unambiguously negative.

## 6. Conclusions

The introduction of the concept of instantaneous rate of growth of consumption in excess of habit seems to be relevant when consumption fully remember habit excess in the previous instant. Even though the assumption  $r = \rho$  can be delimitation, this does not lead to a simplistic dynamics of consumption. Under the logarithmic utility assumption, closed form solutions of the instantaneous rate of growth of consumption in excess of habit, consumption and real money balances were obtained.

By considering in the logarithmic utility the growth rate of consumption in excess of habit, it is found that consumption increases linearly with time due to the habit, while money demand remains constant to finance the habit through time. Also, a once-and-for-all increase in the nominal interest rate results in a reduction of real money balances.

Needlesst to say, the consideration of utility of the growth rate of consumption in excess of habit provides richer environments for studying rational behavior of habit formation with promising future research.



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