ISSN 1870-2171
PANORAMA ECONÓMICO, vol. XIX, núm. 39, julio-diciembre, 2023, pp. 91-109

Asian options as a rational response to post-covid market volatility

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(*Recibido: septiembre,* 2022/*Aceptado: enero,* 2023)

Abstract

In this paper, using a stochastic dynamic general equilibrium model and an economic rationality approach, we maximize a HARA-type utility for a rational economic agent that can use its resources to finance consumption or to invest in a portfolio. By managing its risk, the economic agent avoids losses while hedging his portfolio. The portfolio includes a risk-free bond, a stock, and a long position in an Asian put option whose underlying price is an n-day mean of the stock's price. After ten thousand simulations, we proved that our strategy results in higher portfolio values when compared to other buy-and-hold strategies. In addition, we deducted a valuation formula for the Asian option from the solution process of a differential equations system. The proposed solution is consistent with the Black-Scholes-Merton model.

Keywords: equilibrium models, consumption and portfolio decisions, Asian options.
JEL clasification: D81, D91, G13.

This research was supported by the research projects: "Valuation of Asian-type options on an underlying whose price dynamics have stochastic interest rates and volatility, through economic rationality", SIP code 20221252 and "Calibration and analysis of coupon yield curves cero", SIP code 20221767, both from the Escuela Superior de Economìa of Instituto Politécnico Nacional.

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Las opciones asiáticas como respuesta racional a la volatilidad del mercado post-covid

Resumen

En este artículo, utilizando un modelo de equilibrio general dinámico estocástico y un enfoque de racionalidad económica, maximizamos una utilidad tipo HARA para un agente económico racional que puede usar sus recursos para financiar el consumo o invertir en una cartera. Al gestionar su riesgo, el agente económico evita pérdidas y al mismo tiempo cubre su cartera. La cartera incluye un bono libre de riesgo, una acción y una posición larga en una opción de venta asiática cuyo precio subyacente es una media de n días del precio de la acción. Después de diez mil simulaciones, demostramos que nuestra estrategia genera valores de cartera más altos en comparación con otras estrategias de compra y retención. Además, dedujimos una fórmula de valoración para la opción asiática del proceso de solución de un sistema de ecuaciones diferenciales. La solución propuesta es consistente con el modelo de Black-Scholes-Merton.

Keywords: Palabras clave: modelos de equilibrio, decisiones de consumo y de cartera, opciones asiáticas.
JEL clasificación: D81, D91, G13.

1. Introduction

In recent months, we have experienced economic and market extreme values and volatility that can be considered more aggressive than its observed levels before the COVID-19 epidemic. While facing a new uneven reality, economic agents-investors urgently demand tools to strategically manage the risks generated by the pandemic and now the inflation and market volatility while maximizing their returns.

The immediate response of the financial markets to the pandemic scenario was to focus their efforts on restoring the confidence of the various economic agents in the strength of the markets and their ability to assimilate these extraordinary circumstances. However, as the months went by, extraordinary circumstances became commonplace. The economic agents had to redefine or reinforce their traditional investment tools and strategies, including derivative instruments that allow them to manage a changing, volatile and non-normal environment.

To deal with extraordinary events such as the COVID epidemic or its consequences, under the rational economic agent assumption, economists may use the Stochastic Dynamic General Equilibrium (SDGE). The SDGE is mathematically rooted in the Stochastic Optimum Control (SOC) in a continuous time framework.

Examples of this technology usage are the works of Merton (1971), Venegas-Martínez (2001), (2006) and (2008), Hernández -Lerma (1994), Björk (2009), and Huyên (2009), Martínez-Palacios and Venegas-Martínez (2012) and Martínez-Palacios, Venegas-Martínez and Martínez-Sánchez (2015), among others. Hernández-Lerma (1994) develops economic and financial applications, among others, of controlled Markovian diffusion processes, while Martínez-Palacios, Venegas-Martínez, and Martínez-Sánchez (2015) deduce and characterize the formula for valuing American put options and Björk (2009) presents the optimal stochastic control theory for modeling the optimal portfolio selection and consumption problem.

In this context, widely applicable stochastic processes are the stopping-time functions. Formal foundations of its use appear in Shreve (1997) and Björk (2009), while examples of theoretical-practical applications are Merton (1992), Björk (2009) or Martínez-Palacios, Venegas-Martínez and Martínez-Sánchez (2015). In their paper, they characterize an approximate formula for the valuation of American put options on an underlying asset with stochastic volatility, where a stop-time restriction established the boundary condition corresponding to the intrinsic value of the instrument. On the other hand, Sethi and Thompson (2000) define the stopping-time function as when the economic agent can go bankrupt.

Note that the Asian Options are derivatives contracts that oblige the seller to buy (sell) an underlying asset and give the counterparty the right to sell (buy) the referred asset at a pre-established strike price, K, at a future expiration date, T. The option's value is called the option premium; for more details, see Hull (2014).

In this paper, we use a path-dependent derivative contract called Asian options. Its premium is defined based on an arithmetic or geometric average, which may be discrete or continuous and which, in turn, is distinguished by having an average value of the underlying asset and a fixed strike price or those with an average strike price. The main idea behind using these options is to avoid the false-start signals on American options, or the stop-loss triggered orders generated when the underlying price touches the strike but quickly returns to previous levels (extreme value) or in a volatile environment where the classical European option may be caught in a local minimum that heavily affects the portfolio value.

There is a vast literature regarding option valuation methods; for example, we mention Arregui and Vallejo (2001), Vanmaele *et al*. (2006), Shuguang, Shuiyong and Lijun (2006), Pascucci (2007), Li and Chen (2016), Wang and Zhang (2018). Pirjol and Zhu (2018), and Ocejo (2018). We want to recall the paper from Peng and Peng (2010), who propose a binomial tree method by which they estimate the pricing process when the underlying has Constant Elasticity of Variance (CEV) to price arithmetic-average Asian options.

In the same vein and from any theoretical approach, the valuation of options is relevant to mitigating market risks due to its well-known applicability in hedging, arbitration, and speculation. In particular, the Asian-type options may be related to the widely used Moving Average Convergence Divergence, MACDs, or Estochastich oscillators. The interested practitioner may use those tools to calibrate the moving average length as a function of the asset's volatility and its roofs, channels, and floors; each user may make such decisions based on his beliefs and risk aversion.

We want to emphasize this aspect in the paper. The usage of Asian options is suitable even for hedging medium-sized firms from the commodities price or the exchange rate volatility; both situations have become common in the post-COVID months.

The literature on the subject is wide and varied; consider the seminal articles by Black and Scholes (1973), Merton (1973), Cox and Ross (1976), and Cox, Ingersoll, and Ross (1985a) and (1985b).

In terms of the previous considerations, this paper reviews a protective put. The strategy is recommended in environments of possible growth but with the possibility of falls and volatility. We analyze the protective put from the perspective of dynamic optimization, which implies a long position in the underlying asset, whose price may grow, plus a long put option. The economic agent will exercise the option if the asset's price falls consistently (on an n days arithmetic average) below the strike price. The Asian options ensure that the hedge enters only if the downward trend is confirmed, so it is an alternative to the commonly used stop-loss order.

The best example of this strategy's benefits is the COVID crisis behavior of retirement funds. During the lockdown, the funds suffered significant losses, which reversed when economic activities resumed. Using vanilla put options or stop-loss orders may mean closing positions and realizing potential losses.

Those huge portfolios are not the only firms that may beneficiate from this strategy. In the paper, we show that it is rational for any economic agent, such as medium firms, to use Over The Counter (OTC) Asian put options to cope with extreme values and volatility in exchange rates or relevant commodities.

It is relevant to point out that the model prevents losses to the investor anytime if short sales are allowed and unlimited; this includes sufficient liquidity to deal with margin requirements. In the particular case of the protective put, the asset and the put positions are long, so they do not need additional margins; this is not the case for the counterparts. They must be high credit-quality economic agents.

This paper is distinguished by: 1) Establishing a formula to value Asian options with an arithmetic average underlying price and a fixed exercise price through an approach of economic rationality, consistent with the Black-Scholes model, and 2) Delimit the complete model in the context of economic rationality using a stop-time function, which avoids losses for the investor in the problem's time horizon. 3) Show that the extreme value and volatility clusters, widely extended on the Covid pandemic aftermaths, may be controlled using protective puts.

This work has the following order: in the second section, we state the hypotheses and their corresponding analytical expression; section three analytically shows the inter-temporal budget constraint of the economic agent, while in section four, we define the time horizon using a stopping-time function and Stochastic Dynamic General Equilibrium (SDGE). In the fifth section, we solve the SDGE model by obtaining the partial differential equation (PDE) of Hamilton-Jacobi-Bellman (HJB), from which we optimize the controls that give a HARA-type utility function for consumption.

In the sixth section, we obtain the optimal decisions or optimal controls, for which we propose a solution function in separable variables for the PDE of HJB. In section seven, we assess the model of a European-type Asian put option through a system of differential equations. In the eighth section, we complete the solution process of the PDE-HJB, applying the verification theorem of dynamic programming. In the ninth section we present the portfolio's simulation. Finally, the section tenth establishes conclusions.

2. Hypothesis and its analytical expression

We intend to deduce the formula to assess a European-type Asian put option with an economic rationality approach. For this, we built an SDGE model where the portfolio investment strategy induces economic rationality through hedging a risky asset with the Asian option. We establish the assumptions of the model and the corresponding analytical representation below.

Let us suppose a representative rational economic agent in a small (price-taking) and closed economy with an initial, t=0, wealth, W_o; this economic agent needs to determine the wealth's proportion used for consumption and portfolio that maximizes his utility's present value within the [0, T] time interval without any loss in his portfolio. We assume that there is a banking system with a unique and continuously compounded credit with a risk-free interest rate. We also assume the production and consumption of a single perishable good.

Our economic agent may invest in three assets: a zero-coupon bond priced $B_0 = B(0)$ at $t = 0$. A differential equation drives the bond's value with the form.

$$dB_t = rB_t$$

so the bond investment growth is described as an initial value problem; this is:

$$\frac{dB_t}{dt} - rB_t = 0 \qquad with \qquad B(0) = B_0$$

which implies a yield given by

$$dR_B \equiv \frac{dB_t}{B_t} = rdt \tag{1}$$

the second possible asset is a risky one that corresponds to a common stock whose price is driven by a Markovian-controlled stochastic process

$$dS_t = \mu S_t dt + \sigma S_t dZ_t$$

whose yields are driven by

$$dR_S \equiv \frac{dS_t}{S_t} = \mu dt + \sigma dZ_t \tag{2}$$

Where $0<\mu\in R$ y $\sigma\in R$ represent the average expected yield and its instantaneous volatility, respectively; here Z_t is a Brownian motion defined over a fixed space probability with an augmented filtration $(\Omega, F, (F_t^Z)_{t\in[0,T]}, P)$.

We choose the last portfolio option with hedge porpoises when an asset's price falls. The asset is a long Asian put option. In particular, the Asian option uses as the underlying's price the arithmetic n-day mean on the underlying and a fixed strike price, K. Its intrinsic value is given by $(M_{t,T}$-K,0), where.

$$M_{t,T} = \frac{1}{T-t}\int_t^T S_u\, du$$

Given a fixed temporal horizon, $[0,T]$, we define

$$A_t = \int_0^T S_u\, du \quad \Rightarrow \quad dA_t = S_t dt \tag{3}$$

Please observe that S_t is a price related to the $(F_t^Z)_{t\in[0,T]}$ filtration. Therefore it does not store any history. So, we consider A_t and S_t as independent.

We also want to address the reader's attention to the, r, μ,σ,K y T, contract-fixed parameters. Then the Asian option's price may be denoted as $O_t = O_t$ (S_t, A_t,t). To get the option's yield, we get a stochastic differential equation through Itôs lemma; this is

$$dO_t = \left(\frac{\partial O_t}{\partial t} + \frac{\partial O_t}{\partial S_t}\mu S_t + \frac{\partial O_t}{\partial A_t}S_t + \frac{1}{2}\frac{\partial^2 O_t}{\partial^2 S_t}\sigma^2 S_t^2\right)dt + \frac{\partial O_t}{\partial S_t}\sigma S_t dZ_t \tag{4}$$

where the option's yield is

$$dR_O \equiv \frac{dO_t}{O_t} = \mu_O dt + \sigma_O dZ_t \tag{5}$$

where

$$\mu_O = \frac{1}{O_t}\left(\frac{\partial O_t}{\partial t} + \frac{\partial O_t}{\partial S_t}\mu S_t + \frac{\partial O_t}{\partial A_t}S_t + \frac{1}{2}\frac{\partial^2 O_t}{\partial^2 S_t}\sigma^2 S_t^2\right) \quad y \quad \sigma_o = \frac{1}{O_t}\left(\frac{\partial O_t}{\partial S_t}\sigma S_t\right) \tag{6}$$

Using β_{1t}, β_{2t} and 1-β_{1t}-β_{2t}, we denote the wealth proportion used in the bond, the stock, or the Asian Put. The economic agent uses any remanent as consumption, c_t.

To maintain the model's analytical structure as treatable as possible, we have assumed that all the portfolio strategies are self-financing and, implicitly, the financial's market continuity.

We also assume no transaction costs nor taxes and that markets allow unlimited short-selling. These working assumptions are standard in academic literature, as in Merton (1971, 1973), and correspond to complete markets where it is possible to recreate any derivative, i.e., creating a synthetic. Under the complete market assumption, there is only one neutral risk set of probabilities and thus a unique derivative's price; if the theoretical framework abandons such assumption, there may appear two or more neutral risk probabilities sets and prices, the product of dynamic differential stochastic inclusions; those topics are beyond the reach of this work.

3. Intertemporal budget restriction

In this section, we analytically describe the representative agent's intertemporal budget constraint. We should start by considering that, in the time, *t*, the controlled Markovian process, W_t represents the agent's wealth, this is:

$$\begin{aligned} dW_t &= W_t(1-\beta_{1t}-\beta_{2t})dR_B + W_t\beta_{1t}dR_S + W_t\beta_{2t}dR_O - c_t dt \\ &= W_t\left(r+\beta_{1t}(\mu-r)+\beta_{2t}(\mu_O-r)-\frac{c_t}{W_t}\right)dt + W_t(\beta_{1t}\sigma+\beta_{2t}\sigma_O)dZ_t \end{aligned} \tag{7}$$

in the same manner,

$$\frac{dW_t}{W_t} = \mu_W dt + \sigma_W dZ_t \tag{8}$$

where

$$\mu_W = \left(r+\beta_{1t}(\mu-r)+\beta_{2t}(\mu_O-r)-\frac{c_t}{W_t}\right) \quad y \quad \sigma_W = (\beta_{1t}\sigma+\beta_{2t}\sigma_O) \tag{9}$$

4. Stochastic dynamic general equilibrium (SDGE)

4.1 Stopping time

We must observe that the unlimited short-selling assumption may generate unlimited losses to the economic agent. Mathematically, we can solve the problem by a functional's domain restriction of the form $D=[0,T]\times\{w|w>0\}$ and also defining the function:

$$\tau = min[inf\ \{t>0|W_t=0\}, T]$$

we can economically interpret that restriction as a cease of all economic activity when the wealth process touches the domain's borderline. For more details, see (Björk, 2009).

4.2 The stochastic dynamic general equilibrium model (SDGE)

The agent's intertemporal expected utility function is represented by

$$E\left[\int_0^T F(c_s,s)\,ds|F_0\right]$$

where F represents the consumption utility function. Once we set up the assumptions, we may formally state the utility maximization problem as a continuos-time optimal control problem as

$$\begin{gathered} Maximize_{\beta_{1t},\beta_{2t},c_t}\, E\left[\int_0^{\tau} F(c_s,s)\,dt|F_0\right] \\ dW_t = W_t\mu_W\,dt + W_t\sigma_W\,dZ_t \\ W_0 = w_0 \\ c_t \geq 0, \forall t \geq 0 \end{gathered} \tag{10}$$

5. SDGE model's solution

We can solve the SDGE problem using dynamic programming by reducing the problem's dimension to a deterministic one. We achieve this by getting de Hamilton-Jacobi-Bellman's functional. Departing on that functional, we can optimize the controls and, therefore, the model's optimal choices; this is

$$\begin{aligned} J(W_t,A_t,t) &= \max_{\beta_{1t},\beta_{2t}\in R,0\leq c_s|_{[t,\tau]}} E\left[\int_t^{\tau} F(c_s,s)\,ds|F_t\right] \\ &= \max_{\beta_{1t},\beta_{2t}\in R,0\leq c_s|_{[t,\tau]}} E\left[\int_t^{t+dt} F(c_s,s)\,ds + \int_{t+dt}^{\tau} F(c_s,s)\,ds|F_t\right] \end{aligned} \tag{11}$$

By applying continuous time dynamic programming to (11), we can deduct the HJB equation on which we impose the model's border conditions; this is

$$\begin{aligned} \{0 = \max_{\beta_{1t},\beta_{2t}\in R,0\leq c_t} &\left\{F(c_t,t) + \frac{\partial J(W_t,A_t,t)}{\partial t} + \frac{\partial J(W_t,A_t,t)}{\partial W_t}W_t\mu_W + \frac{\partial J(W_t,A_t,t)}{\partial A_t}S_t \right. \\ &\left. + \frac{1}{2}\frac{\partial^2 J(W_t,A_t,t)}{\partial W_t^2}W_t^2\sigma_W^2\right\} \end{aligned} \tag{12}$$

$$J(0,A_t,t,) = 0$$

$$J(W_t,A_t,T) = 0\}$$

5.1 Utility function

We propose the utility function given by $F(^{ct},t)=e^{-\rho t}\,V(ct) = e^{-\rho t}\,c^{\gamma}$, $0<\gamma<1$, where $V(c_t)$ is a Hyperbolic Absolute Risk Aversion (HARA) type function (Merton, 1990 y Hakansson, 1970) to find analytical solutions to our problem. It is necessary to observe that $V(c_t)$ has the following property.

$$V'(0) = \gamma \frac{c^{\gamma}}{c} |_{c=0} = \infty$$

therefore, the solution must meet the non-negative condition on the portfolio.

5.2 First order conditions

In equation (12), we substitute μ_W, σ_W y $F(c_t, t)$, to get

$$0 = e^{-\rho t} c^{\gamma} \frac{\partial J(W_t, A_t, t)}{\partial t} + \frac{1}{2} \frac{\partial^2 J(W_t, A_t, t)}{\partial W_t^2} W_t^2 (\beta_{1t}\sigma + \beta_{2t}\sigma_O)^2$$
$$\frac{\partial J(W_t, A_t, t)}{\partial W_t} W_t \left(r + \beta_{1t}(\mu - r) + \beta_{2t}(\mu_O - r) - \frac{c_t}{W_t} \right) + \frac{\partial J(W_t, A_t, t)}{\partial A_t} S_t \tag{13}$$

Using the first-order criteria, given a concave function, we optimize the control variable β_{1t}, β_{2t} y c_t.

$$c_t = \left[\frac{\partial J(W_t, A_t, t)}{\partial W_t} \frac{e^{\rho t}}{\gamma} \right]^{\frac{1}{\gamma - 1}} \tag{14}$$

$$\beta_{1t} = - \frac{\frac{\partial J(W_t, A_t, t)}{\partial W_t} W_t (\mu - r) + \frac{\partial^2 J(W_t, A_t, t)}{\partial W_t^2} W_t^2 \sigma \beta_{2t} \sigma_O}{\frac{\partial^2 J(W_t, A_t, t)}{\partial W_t^2} W_t^2 \sigma^2} \tag{15}$$

$$\beta_{2t} = - \frac{\frac{\partial J(W_t, A_t, t)}{\partial W_t} W_t (\mu_O - r) + \frac{\partial^2 J(W_t, A_t, t)}{\partial W_t^2} W_t^2 \sigma \beta_{1t} \sigma_O}{\frac{\partial^2 J(W_t, A_t, t)}{\partial W_t^2} W_t^2 \sigma_O^2} \tag{16}$$

6. Optimal decisions

To solve the Differential Partial Equation (DPE) given by the Hamilton Jacobi Bellman, we need to establish a $J(W_t, A_t, t)$ functional. Given the differential equation nature, we propose a solution candidate, J, as a variable-separable equation of the form:

$$J(W_t, A_t, t) = e^{-\rho t} f(A_t, t) W_t^{\gamma}, \quad f(A_t, T) = 0 \tag{17}$$

We impose a $f(A_t, T)=0$ boundary condition to (17), corresponding to equation (12). Given the proposed solution, $J(W_t, A_t, t)$.We got the partial derivatives expressed on the Hamilton – Jacobi – Bellman equation.

$$\frac{\partial J(W_t, A_t, t)}{\partial \mathrm{t}} = W_t^{\gamma} e^{-\rho t} \frac{\partial f(A_t, t)}{\partial t} - \rho X_t^{\gamma} e^{-\rho t} f(A_t, t)$$
$$\frac{\partial J(W_t, A_t, t)}{\partial A_t} = W_t^{\gamma} e^{-\rho t} \frac{\partial f(A_t, t)}{\partial A_t}$$
$$\frac{\partial J(W_t, A_t, t)}{\partial W_t} = \gamma W_t^{\gamma-1} e^{-\rho t} f(A_t, t) \tag{18}$$
$$\frac{\partial^2 J(W_t, A_t, t)}{\partial W_t^2} = \gamma(\gamma - 1) W_t^{\gamma-2} e^{-\rho t} f(A_t, t)$$

The next step is to substitute (18) in (14), (15), and (16). After that substitution, we may find the problem's optimal controls.

$$\hat{c}_t = W_t [f(A_t, t)]^{\frac{1}{\gamma-1}} \tag{19}$$

$$\hat{\beta}_{1t} = -\frac{(\mu - r) + (\gamma - 1)\sigma\beta_{2t}\sigma_O}{(\gamma - 1)\sigma^2} \tag{20}$$

$$\hat{\beta}_{2t} = -\frac{(\mu_O - r) + (\gamma - 1)\sigma\beta_{1t}\sigma_O}{(\gamma - 1)\sigma_O^2} \tag{21}$$

Please observe that (20) and (21) constitute a stochastic partial differential equations system,

$$\begin{cases} \hat{\beta}_{1t} + \dfrac{\hat{\beta}_{2t}\sigma_O}{\sigma} = -\dfrac{\mu - r}{(\gamma - 1)\sigma^2} \\ \\ \hat{\beta}_{2t} + \dfrac{\hat{\beta}_{1t}\sigma}{\sigma_O} = -\dfrac{\mu_O - r}{(\gamma - 1)\sigma_O^2} \end{cases} \tag{22}$$

If we consider the following variable changes, $\xi = \frac{\sigma_O}{\sigma}$, $\lambda = \frac{\mu - r}{(1-\gamma)\sigma^2}$ and $\lambda_O = \frac{\mu_O - r}{(1-\gamma)\sigma_O^2}$ In (22), we get.

$$\left.\begin{aligned} \hat{\beta}_{1t} + \xi\hat{\beta}_{2t} &= \lambda \\ \frac{\hat{\beta}_{1t}}{\xi} + \hat{\beta}_{2t} &= \lambda_O \end{aligned}\right\} \quad \Rightarrow \quad \begin{pmatrix} 1 & \xi \\ \xi^{-1} & 1 \end{pmatrix} \begin{pmatrix} \hat{\beta}_{1t} \\ \hat{\beta}_{2t} \end{pmatrix} = \begin{pmatrix} \lambda \\ \lambda_O \end{pmatrix} \tag{23}$$

Observe that the coefficients' determinant matrix is zero; therefore, there is lineal dependence between the risk-premium equations in the stock and derivative markets.

7. A model for a European-style Asian option

The linear dependence in (23) allows us to establish the following equality

$$\lambda = \xi\lambda_O \quad \Rightarrow \quad \frac{\mu - r}{(1-\gamma)\sigma^2} = \frac{\sigma_O}{\sigma} \frac{\mu_O - r}{(1-\gamma)\sigma_O^2}$$

which drives to

$$(\mu - r)\frac{\sigma_O}{\sigma} = (\mu_O - r)$$

After substituting in the last equation μ_O and σ_O As described in (6), we get

$$\frac{\partial O_t}{\partial t} + \frac{\partial O_t}{\partial A_t} S_t + \frac{1}{2}\frac{\partial^2 O_t}{\partial^2 S_t}\sigma^2 S_t^2 + \frac{\partial O_t}{\partial S_t} S_t r - rO_t = 0 \tag{24}$$

Equation (24) is the model that allows us to assess the European-type Asian option; this partial differential equation has an infinite curve family as a solution. To get a unique solution that characterizes the Asian option with an underlying given by the arithmetic *n*- days asset's price means allows us to optimize the portfolio while minimizing the investor's risk derived from the covid's aftermath volatility. We must impose a borderline condition with the intrinsic derivative's value at the expiration date; this is:

$$\begin{cases} \dfrac{\partial O_t}{\partial t} + \dfrac{\partial O_t}{\partial A_t} S_t + \dfrac{1}{2}\dfrac{\partial^2 O_t}{\partial^2 S_t}\sigma^2 S_t^2 + \dfrac{\partial O_t}{\partial S_t} S_t r - rO_t = 0 \\ O(S_t, A_t, T) \geq \left(\dfrac{A_T}{T} - K, 0\right) \\ t \leq \underline{\tau} \end{cases} \tag{25}$$

Where K denotes the exercise price of the American option in the exercise time, $\underline{\tau} = \{\tau, \hat{\tau}\}$ where $\hat{\tau}$ is a stopping time such that the portfolio reaches $\left(\frac{A_T}{T} - K, 0\right)$.

8. Verification theorem for the Hamilton-Jacobi-Bellman's partial differential equation

Observe that the optimal controls depend on the proposed solution, the $f(A_t, t)$ function, to the problem stated as the functional $J(W_t, A_t, t)$. So we must verify that equation (17) solves the problem stated in (12). For doing so, we assume a corner solution for the wealth proportion assignation $\hat{\beta}_{1t} = 0$ and $\hat{\beta}_{2t} = 1$, for the common stock and the derivative. We want to emphasize that, because of the linear dependence, there is an infinite set of linear combinations between the portfolio and the option that are optimal controls; this allows any investor to use any strategy to cover or enhance their portfolio. In particular, given the post-COVID volatility, we chose a protective put.

For convenience, we substitute in (13) the equations (6), (19) and, and$\hat{c}_t, \hat{\beta}_{1t} = 0$ *and* $\hat{\beta}_{2t} = 1$ the correspondent partial derivatives in μ_O and σ_O assessed in an at-the-money option, this is: $S_t = K_{t,T}$.

$$\mu_O|_{S_t=K_{t,T}} = \bar{\mu}_O \quad and \quad \sigma_O|_{S_t=K_{t,T}} = \bar{\sigma}_O$$

we got:

$$0 = f(A_t,t)\left((-\rho) + \gamma\bar{\mu}_O + \frac{\gamma}{2}(\gamma-1)\bar{\sigma}_O^2\right) + \frac{\partial f(A_t,t)}{\partial t} + \frac{\partial f(A_t,t)}{\partial A_t}S_t + (\gamma-1)f(A_t,t)^{\frac{\gamma}{\gamma-1}} \tag{26}$$

which, in concordance with (12), we must use the border condition given by $f(A_t, T)=0$.

Please, observe that the solution given by (26) does not depend on A_t. Let us define $k_1 = (-\rho) + \gamma\bar{\mu}_O + \frac{\gamma}{2}(\gamma-1)\bar{\sigma}_O^2$ to the constant in $f(t)$, obtaining the following first degree Bernoulli type partial differential equation

$$f'(t) + k_1 f(t) = (1-\gamma)f^{\frac{\gamma}{\gamma-1}}(t), \qquad f(T) = 0 \tag{27}$$

With the parameters given by $p(t) = k_1, q(t) = 1-\gamma$ and $n = \dfrac{\gamma}{\gamma-1}$.

To solve the Bernoulli equation, we propose the variable change given by $y = f^{1-n}(t) = f^{-\frac{1}{\gamma-1}}(t)$, which implies that $f(t) = y^{1-\gamma}$ and. After substituting in (27) and multiplying both sides of the equation by $\frac{y^\gamma}{(1-\gamma)}$, we get:

$$y' + \frac{k_1}{(1-\gamma)}y = 1, \quad y(T) = 0 \tag{28}$$

equation (28) corresponds to an initial value problem whose differential equation is solved with the integrant factor method; this is

$$\mu(t) = e^{\int \frac{k_1}{(1-\gamma)}dt} = e^{\frac{k_1}{(1-\gamma)}t}$$

Therefore, $y(t)$ is

$$y(t) = \frac{\int 1e^{\frac{k_1 t}{1-\gamma}}dt + k_2}{e^{\frac{k_1 t}{1-\gamma}}} = \frac{\left(\frac{1-\gamma}{k_1}\right)\int e^u du + k_2}{e^{\frac{k_1 t}{1-\gamma}}} = \frac{\left(\frac{1-\gamma}{k_1}\right)e^{\frac{k_1 t}{1-\gamma}} + k_2}{e^{\frac{k_1 t}{1-\gamma}}} = \frac{1-\gamma}{k_1} + k_2 e^{-\frac{k_1 t}{1-\gamma}}, \tag{29}$$

By imposing the $y(T) = 0$ condition to y, we get

$$y(t) = \frac{1-\gamma}{k_1}\left(1 - e^{\frac{k_1}{1-\gamma}(T-t)}\right) \tag{30}$$

Once we got $y(t)$, we substitute this in $f(t)$ and $f'(t)$; this implies $f(t) = y^{1-\gamma}$ y $f'(t) = (1-\gamma)y^{-\gamma}y'$, which, in turn, establish:

$$f(t) = y^{1-\gamma}(t) = \left[\frac{1-\gamma}{k_1}\left(1 - e^{\frac{k_1}{1-\gamma}(T-t)}\right)\right]^{1-\gamma}, \quad f(T) = 0 \tag{31}$$

We can observe that the functional's, J, structure (17) solves (31). So, we can estate that the functional, J, satisfice the Hamilton-Jacobi-Bellman equation, and $\hat{\beta}_{1t}, \hat{\beta}_{2t}$ *and* $\hat{c}$ and $\hat{c}$ are the optimal controls for the maximization problem without having financial losses over the optimization period, even in the volatility post-COVID environment.

9. Portfolio's simulation

The COVID pandemic market losses created actual scenarios that overpass the previous stress-test scenarios that included 2009, the dot-com crises, or any other previous scenario commonly used for the stress test while managing market portfolios; this includes retirement and pension funds or commodity prices affecting the whole logistic chain for a wide variety of industries.

During the COVID pandemic and its aftermaths, even the most prominent companies suffered from unexpected rises in commodities costs, showing that market risk management is a valuable tool not solely for banks or investment companies but for the whole economy. COVID showed us the shortcomings of a cost-minimizing value chain and its dependence on some key commodity prices, all of which are widely tradeable, and interested companies may find hedging instruments as derivatives in the regulated market or Over The Counter their banks. Each company should define its proper strategy's position by analyzing if they are holding (long protective put) or needing the asset (short position). We want to emphasize that this is not the only hedging strategy and that our results are adaptable for most derivatives as long as they satisfy the abovementioned conditions.

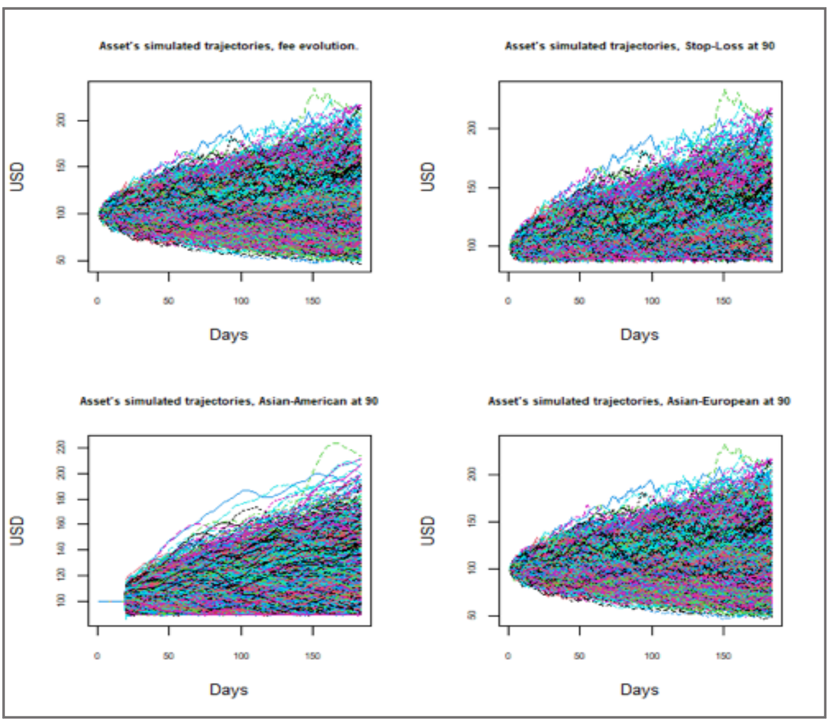
With the idea of testing the American-Asian option's performance in a portfolio, we carried on ten thousand simulations on a set of portfolios:

1. Buy-and-hold an asset-only portfolio: There are no additional costs in this strategy.
2. An asset-only portfolio with a stop-loss order, trigger at K USD. In this case, the investor should buy the asset and, at the same time, set

a sell order in case the asset's price touches or falls below the trigger price. There are no additional costs in this strategy.

3. A protective Euro-Asian put strategy, with a strike price at *K* USD. In this case, the investor should buy the asset and a long Euro-Asian put. If the n-day asset's average price touches or falls below the strike, *K*, at the expiration date, the investor receives *K*. This strategy must cover the option's premium, P_{EA}.
4. A protective American-Asian put strategy, with a strike price at *K* USD. In this case, the investor should buy the asset and a long American-Asian put. If the n-day asset's average price touches or falls below the strike, *K*, at any time, the investor receives *K*. This strategy must cover the option's premium, P_{AA}.

Graph 1
Portfolio's simulated paths with different strategies.
Own elaboration with *R*



Fuente: elaboración propia.

Graph 1 shows [2] the general simulated paths, with a strike price of 90 USD, to make an example of the simulation process. We can see the evolution of the portfolio under the same asset paths and different investment strategies.

As the reader may observe, the first strategy may result in a profound loss in the portfolio´s value, but if the asset price drop occurs before the expiration date, the portfolio may recover its value.

As a result of the second strategy, the stop-loss, the portfolio ends its path in *K* if it reaches that price even for a moment, dropping any chance of recovery.

The third strategy, the Euro-Asian protective put, allows the portfolio to recover from sudden drops if there is enough time on its time path but may be slow to detect changes in the trend or any other memory effects not simulated in our paper. Having a too long n- days average may worsen the loss effect.

Finally, the fourth strategy, the American-Asian protective put, allows the portfolio to recover from those sudden losses but allows it to cut the downward path if the n-day average confirms that tendency. This strategy fits with the most common tools used in technical analysis.

The following table, table 1, shows the simulated results for different strike prices, *K*, for the four strategies. We want to stress that they all use the same paths for the asset´s price to focus on the strategy's results. In all cases, the simulated initial asset's price is 100 USD, and the daily simulation lasts for 180 days following a stochastic yield given in (2)

Table 1
Simulation results under different portfolio options

Strategy/K	**90**	**70**	**50**	**99**
1)	104.3039	103.9542	104.3557	104.1135
2)	103.2292	103.9435	104.3572	101.6642
3	104.6334	104.0195	104.3570	104.4796
4)	104.9915	103.8743	104.2045	105.0193

Own: elaboration with *R*

[2] The used code is available on demand at the author's e-mail.

The reader may observe that the American-Asian strategy gives the best results as the strike price is closer to the initial one; this is when there is uncertainty about the future and resistance, or floor, is below the current asset's price, and there are some recovery signals; this was the economic scenario after the COVID pandemic, just after the economic reopening.

10. Conclusions

This paper demonstrates the economic rationality behind a commonly used investment strategy in the market. We demonstrate that a technical analysis-based strategy, the protective put, has an economically based rationale.

We also demonstrate the valuation equation to assess the European-style Asian put options, with the underlying price given by an n-day average price and fixed strike. We demonstrate that the option is consistent with the Black-Scholes-Merton (BSM) model.

The reader may observe that the dynamic programming verification theorem led us to an analytic solution that solves the partial differential equation; this demonstrates that we may mitigate the portfolio risk using the optimal controls (portfolios' investment proportions) to maximize the utility function along the time horizons.

We also want to point out that the proposed solution forbids financial losses to the economic agent. Analytically, we achieve the non-loosing condition by implementing a Hyperbolic Absolute Risk Aversion, HARA, type of utility function that narrows all the possible strategies to non-loosing paths over the utility's function time, hence introducing rationality to the model and the whole maximization process.

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